On the problem of uniqueness of energy-momentum tensor of gravitational field

A.I.Nikishov *

I.E.Tamm Department of Theoretical Physics,
P.N.Lebedev Physical Institute, Moscow, Russia

February 7, 2008

Abstract

For an island-like distribution of matter the gravitational energy-momentum tensor is defined according to Weinberg as a source of metric. If this source is formed by self-interactions of gravitons, so that nonphysical degrees of freedom are excluded, then this source is a reasonable candidate for the energy-momentum tensor of gravitational field. The disastrous influence of the nonphysical degrees of freedom is demonstrated by comparing the gravitational energy-momentum tensors in the harmonic, isotropic and standard frames for the Schwarzschild solution. The harmonic frame is clearly preferable for defining the gravitational energy-momentum tensor.

1 The gravitational energy-momentum tensor as the source of metric

There are several arguments in favor of non-localizability of the energy of the gravitational field, see §20.4 in [1]. They do not seem convincing enough.

Following [2], we consider the case when

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \tag{1}$$

and $h_{\mu\nu} \to 0$ quickly enough when $x \to \infty$, but it is not assumed that $h_{\mu\nu} \ll 1$ everywhere. The wave equation for $h_{\mu\nu}$ is

$$h_{\mu\nu,\lambda}{}^{\lambda} - h^{\lambda}{}_{\mu,\lambda\nu} - h^{\lambda}{}_{\nu,\lambda\mu} + h_{,\mu\nu} + \eta_{\mu\nu}(h_{\sigma\lambda}{}^{,\sigma\lambda} - h_{,\lambda}{}^{\lambda}) =$$

$$-16\pi G(T_{\mu\nu} + t_{\mu\nu}), \quad h \equiv h_{\lambda}{}^{\lambda}; \quad h_{,\sigma} \equiv \frac{\partial}{\partial x^{\sigma}}, \tag{2}$$

cf. Ch.3, §17 in [3]. Here $t_{\mu\nu}$ is the gravitational energy-momentum tensor.

*E-mail: nikishov@lpi.ru

In general relativity eq. (2) is the Einstein equation with

$$8\pi G t_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} R^{(1)\lambda}{}_{\lambda}, \tag{3}$$

see equations (7.6.3) and (7.6.4) in [2]. Indices in $h_{\mu\nu}$, $R^{(1)}_{\mu\nu}$ and $\frac{\partial}{\partial x^\sigma}$ are raised and lowered with the help of η, while indices of true tensors, such as $R_{\mu\nu}$ are raised and lowered with the help of g as usual. $R^{(1)}_{\mu\nu}$ is a linear in $h_{\mu\nu}$ part of $R_{\mu\nu}$:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}[h_{,\mu\nu} - h^{\lambda}{}_{\mu,\lambda\nu} - h^{\lambda}{}_{\nu,\lambda\mu} + h_{\mu\nu,\lambda}{}^{\lambda}], \quad h = \eta^{\mu\nu} h_{\mu\nu} \quad , \quad R^{(1)} = \eta^{\mu\nu} R^{(1)}_{\mu\nu}. \tag{4}$$

As shown in [2], tensor (3) has all the necessary properties of a gravitational energy- momentum tensor. The same is true even if the general relativity is not assumed i.e. $t_{\mu\nu}$ has not the form (3). Non the less, $t_{\mu\nu}$ as defined in (3) has one drawback: it depends on a coordinate system in an inadmissible way. I shall demonstrate this for the gravitational field of a spherically symmetric body and then indicate how, in my opinion, to correct the situation.

To begin with, we write down some of metrics outside the body.

1. Standard Schwarzschild

$$d\tau^2 = (1 - \frac{2GM}{r})dt^2 - r^2(\sin^2\theta d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{2GM}{r}}. \tag{5}$$

2. Harmonic ($R = r - GM$)

$$d\tau^2 = \frac{1 - \frac{GM}{R}}{1 + \frac{GM}{R}} dt^2 - \left(1 + \frac{GM}{R}\right)^2 R^2(\sin^2\theta d\varphi^2 + d\theta^2) - \frac{1 + \frac{GM}{R}}{1 - \frac{GM}{R}} dR^2. \tag{6}$$

3. Isotropic ($r = \rho(1 + \frac{GM}{\rho})^2$)

$$d\tau^2 = \left(\frac{1 - \frac{GM}{2\rho}}{1 + \frac{GM}{2\rho}}\right)^2 dt^2 - \left(1 + \frac{GM}{2\rho}\right)^4 [d\rho^2 + \rho^2(\sin^2\theta d\varphi^2 + d\theta^2)]. \tag{7}$$

4. Eddington frame

$$d\tau^2 = dt^{*2} - \frac{2GM}{r^*}(dr^* + dt^*)^2 - dr^{*2} - r^{*2} d\Omega. \tag{8}$$

It is remarkable that in this frame $g_{\mu\nu}$ linearly depends upon G. Although all frames are equal, at least for our purpose there is a frame that is "more equal than others".

Now we go back to equation (2) and consider space region where $T_{\mu\nu} = 0$. Then the equation tell us that if we know $h_{\mu\nu}$, we can obtain $t_{\mu\nu}$. Using this, we get the gravitational energy-momentum tensors in the harmonic, isotropic and standard Schwarzschild frames. Instead of spherical coordinates in (5-7) we use below the rectangular ones and denote the space coordinates by x_i, $\quad x_i x_i = r^2$, $\quad i = 1, 2, 3$ *in all three frames.*

2 The harmonic frame

In this frame the gravitational energy-momentum tensor was obtained in [4]. We have

$$h_{00} = \frac{-2\phi}{1-\phi}, \quad h_{ij} = (-2\phi + \phi^2)\delta_{ij} + \frac{1-\phi}{1+\phi}\phi^2 \frac{x_i x_j}{r^2}, \quad h_{0i} = 0, \quad \phi = -\frac{GM}{r}. \tag{9}$$

$$8\pi G t_{00}^{har} = \frac{1}{r^2}[-2 - \phi^2 + \frac{4}{1+\phi} - \frac{2}{(1+\phi)^2}], \qquad t_{0i}^{har} = 0; \tag{10}$$

$$8\pi G t_{ij}^{har} = \frac{x_i x_j}{r^4}\left(-2\phi^2 + 2 - \frac{1}{1+\phi} - \frac{1}{(1+\phi)^2} + \frac{1}{1-\phi} + \frac{1}{(1-\phi)^2} - \frac{2}{(1-\phi)^3}\right) +$$
$$\frac{\delta_{ij}}{r^2}\left(\phi^2 - \frac{1}{1+\phi} + \frac{1}{(1+\phi)^2} + \frac{1}{1-\phi} - \frac{3}{(1-\phi)^2} + \frac{2}{(1-\phi)^3}\right), \quad i,j = 1,2,3. \tag{11}$$

We see that the energy-momentum tensor has singularities at $r = GM$, when $\phi = -1$.

For $\phi \ll 1$ we get

$$8\pi G r^2 t_{00}^{har}\big|_{\phi\ll 1} = -3\phi^2 + 4\phi^3 + \cdots; \tag{12}$$

$$8\pi G t_{ij}^{har}\big|_{\phi\ll 1} = 7\frac{\phi^2}{r^2}\left(\delta_{ij} - \frac{2x_i x_j}{r^2}\right) + 6\frac{\phi^3}{r^2}\left(\delta_{ij} - \frac{5}{3}\frac{x_i x_j}{r^2}\right) + \cdots. \tag{13}$$

Defining $\tilde{h}^{\mu\nu}$ by

$$\tilde{g}^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu}, \quad g = \det g_{\mu\nu}, \tag{14}$$

we get the harmonic condition in the form

$$\tilde{h}^{\mu\nu}{}_{,\nu} = 0. \tag{15}$$

This condition is the analogue of Lorentz condition in electrodynamics. It exclude the nonphysical degrees of freedom and the simplest assumption is that in our case it exclude all the nonphysical degrees. In this case the rectangular harmonic frame is the preferred system, the use of which has been advocated by Fock [5]. In general relativity we have

$$\tilde{g}^{ik} = \delta_{ik} - \phi^2 \frac{x_i x_k}{r^4}, \tag{16}$$

see §58 in [5]. This expression can be obtained also from the second order of perturbation expansion, considered in [6] using quantum tree graphs.

If general relativity is not assumed, we have instead of (16)

$$\tilde{g}^{ik} = \delta_{ik} - \alpha\phi^2 \frac{x_i x_k}{r^4}, \tag{17}$$

where α is a coefficient of order one. It is determined by a chosen 3 graviton vertex. The higher terms in the expansion (17), i.e. terms with ϕ^n and $\phi^n \frac{x_i x_k}{r^2}$, $n > 2$ are still absent due to (15) as seen from the relations

$$\left(\frac{1}{r^n}\right)_{,j} = -\frac{n x_j}{r^{n+2}}, \quad \left(\frac{x_i x_j}{r^n}\right)_{,j} = \frac{x_i(4-n)}{r^n}.$$

Only term with $n = 4$, i.e. term $\phi^2 \frac{x_i x_k}{r^4} = G^2 M^2 x_i x_k / r^4$ is possible.

Using (17) and letting g_{00} to be arbitrary for the time being, we can express g_{ik} through it. First, from (17) we find

$$\det \tilde{g}^{ik} = 1 - \alpha\phi^2. \tag{18}$$

Then we have

$$\det \tilde{g}^{\mu\nu} = \tilde{g}^{00} \det \tilde{g}^{ik} = g. \tag{19}$$

Then, using $\tilde{g}^{00} = \sqrt{(-g)} g^{00} = \sqrt{-g}/g_{00}$, we obtain from (18) and (19)

$$\sqrt{-g} = -\frac{1 - \alpha\phi^2}{g_{00}}.$$

From definition $\tilde{g}^{ik} \tilde{g}_{kj} = \delta_{ij}$ we find

$$\tilde{g}_{kj} = \delta_{kj} + \frac{\alpha\phi^2}{1 - \alpha\phi^2} \frac{x_k x_j}{r^2}. \tag{20}$$

Finally, we have

$$g_{ik} = \sqrt{-g}\tilde{g}_{ik} = -\frac{1}{g_{00}}[(1 - \alpha\phi^2)\delta_{ik} + \alpha\phi^2 \frac{x_i x_k}{r^3}]. \tag{21}$$

Assuming $g_{00} = -\exp(2\phi)$, obtained in [7] from heuristic considerations, we get $g_{\mu\nu}$ and $t_{\mu\nu}$ regular everywhere except $r = 0$. The same form of g_{00} appears in general relativity for a model of a spherical body considered in the cylindrical coordinates, see eq. (8.30) in [8].

In any case, from perihelion precession we know that in G^2 approximation $g_{00} = -(1 + 2\phi + 2\phi^2)$. Then (21) in this approximation gives

$$g_{ik}^{(2)} = [1 - 2\phi + (2 - \alpha)\phi^2] + \alpha\phi^2 \frac{x_i x_k}{r^2}. \tag{22}$$

In general relativity $\alpha = 1$. If we want to preserve the coordinate condition (15) in a more general approach, $g_{ik}^{(2)}$ still must have the form (22).

3 Isotropic frame

In this frame

$$h_{00} = \frac{4}{1 - \frac{\phi}{2}} - \frac{4}{(1 - \frac{\phi}{2})^2}, \quad h_{ij} = \delta_{ij}[-2\phi + \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 + \frac{1}{16\phi^4}], \quad h_{0i} = 0. \tag{23}$$

Here ϕ has the same form as in (9). From here we get

$$h_{ij,km} = \delta_{ij} \frac{x_k x_m}{r^4}[-6\phi + 12\phi^2 - \frac{15}{2}\phi^3 + \frac{3}{2}\phi^4] + + \frac{\delta_{ij}\delta_{km}}{r^2}[2\phi - 3\phi^2 + \frac{3}{2}\phi^3 - \frac{1}{4}\phi^4], \tag{24}$$

$$h_{00,ij} = \frac{\delta_{ij}}{r^2}[\frac{4}{1 - \frac{\phi}{2}} - \frac{12}{(1 - \frac{\phi}{2})^2} + \frac{8}{(1 - \frac{\phi}{2})^3}] + \frac{x_i x_j}{r^4}[-\frac{4}{1 - \frac{\phi}{2}} - \frac{4}{(1 - \frac{\phi}{2})^2} + \frac{32}{(1 - \frac{\phi}{2})^3} - \frac{24}{(1 - \frac{\phi}{2})^4}]. \tag{25}$$

and

$$h=-\frac{4}{1-\frac{\phi}{2}}+\frac{4}{(1-\frac{\phi}{2})^2}+3[-2\phi+\frac{3}{2}\phi^2-\frac{1}{2}\phi^3+\frac{1}{16}\phi^4], \tag{26}$$

Using these expressions in (4), we find

$$R^{(1)}_{00}=\frac{1}{2}h_{00,ii}=\frac{1}{r^2}[\frac{4}{1-\frac{\phi}{2}}-\frac{20}{(1-\frac{\phi}{2})^2}+\frac{28}{(1-\frac{\phi}{2})^3}--\frac{12}{(1-\frac{\phi}{2})^4}], \tag{27}$$

$$R^{(1)}_{ij}=\frac{x_ix_j}{r^4}[\frac{2}{1-\frac{\phi}{2}}+\frac{2}{(1-\frac{\phi}{2})^2}-\frac{16}{(1-\frac{\phi}{2})^3}+\frac{12}{(1-\frac{\phi}{2})^4}-3\phi+6\phi^2-\frac{15}{4}\phi^3+\frac{3}{4}\phi^4]+$$
$$\frac{\delta_{ij}}{r^2}[-\frac{2}{1-\frac{\phi}{2}}+\frac{6}{(1-\frac{\phi}{2})^2}-\frac{4}{(1-\frac{\phi}{2})^3}+\phi-\frac{3}{4}\phi^3+\frac{1}{4}\phi^4] \tag{28}$$

Finally, we obtain from (3) (the first two terms on the r.h.s. of (3) disappear because we use the exact solution of the Einstein equation and consider the region of space without matter)

$$8\pi Gr^2t^{iso}_{00}=-3\phi^2+3\phi^3-\frac{3}{4}\phi^4, \quad \phi=-\frac{GM}{r}, \quad t^{iso}_{0i}=0; \tag{29}$$

(this is an exact expression for t^{iso}_{00}) and similarly

$$8\pi Gt^{iso}_{ij}=\frac{x_ix_j}{r^4}[-\frac{2}{1-\frac{\phi}{2}}-\frac{2}{(1-\frac{\phi}{2})^2}+\frac{16}{(1-\frac{\phi}{2})^3}-\frac{12}{(1-\frac{\phi}{2})^4}+3\phi-6\phi^2+\frac{15}{4}\phi^3-\frac{3}{4}\phi^4]+$$
$$\frac{\delta_{ij}}{\rho^2}[-\frac{2}{1-\frac{\phi}{2}}+\frac{14}{(1-\frac{\phi}{2})^2}-\frac{24}{(1-\frac{\phi}{2})^3}+\frac{12}{(1-\frac{\phi}{2})^4}-\phi+3\phi^2-\frac{9}{4}\phi^3+\frac{1}{2}\phi^4]. \tag{30}$$

For $\phi \ll 1$ we have

$$8\pi Gt^{iso}_{ij}\big|_{\phi\ll 1}=7\frac{\phi^2}{r^2}\left(\delta_{ij}-2\frac{x_ix_j}{r^2}\right)+\frac{9}{2}\frac{\phi^3}{r^2}\left(\delta_{ij}-\frac{5}{3}\frac{x_ix_j}{r^2}\right)+\cdots \tag{31}$$

We note now that $t^{iso}_{\mu\nu}$ is regular everywhere, except at $r=0$. This can be expected because the metric in (7) is regular. The transformation from the harmonic frame to the isotropic one is also regular transformation (as well near horizon): $R=\rho(1+\frac{GM}{2\rho})^2-GM$. So there is no reason for disappearance of singularities in $t^{iso}_{\mu\nu}$. If we assume that $h^{har}_{\mu\nu}$ is formed only by the physical degrees of freedom, we may consider $t^{har}_{\mu\nu}$ as a correct tensor and interpret the disappearance of singularity in $t^{iso}_{\mu\nu}$ as foul play of nonphysical degrees of freedom.

From (12-13) and (29), (31) we see that in ϕ^2 approximation $t^{har}_{\mu\nu}$ coincides with $t^{iso}_{\mu\nu}$. This is in agreement with the fact that $h^{(2)iso}_{\mu\nu}$ (i.e. $h^{iso}_{\mu\nu}$ in the ϕ^2 approximation) can be obtained from $h^{(2)har}_{\mu\nu}$ by gauge transformation

$$h^{(2)har}_{ij}-h^{(2)iso}_{ij}=\frac{1}{2}G^2M^2(\frac{\delta_{ij}}{r^2}-\frac{2x_ix_j}{r^4})=\frac{1}{4}G^2M^2(\Lambda_{i,j}+\Lambda_{j,i}), \quad \Lambda_i=\frac{x_i}{r^2}. \tag{32}$$

Here

$$h^{(2)har}_{ij}=\phi^2\left(\delta_{ij}+\frac{x_ix_j}{r^2}\right), \quad h^{(2)iso}_{ij}=\frac{3}{2}\phi^2\delta_{ij},$$

see (9) and (23). The gauge transformation does not changes the source and may be interpreted as a change of frame, but *not visa versa*. The linear approximation in $h_{\mu\nu}$, i.e. $h^{(1)}_{\mu\nu}$ produces the ϕ^2 approximation in the source (see equation (7.6.15) in [2]) and that is why $t^{(2)}_{\mu\nu}$ coincide in both frames.

4 Standard frame

In this case we have

$$h_{00} = -2\phi, \quad h_{ij} = \frac{x_i x_j}{r^2}\left(\frac{1}{1+2\phi} - 1\right). \tag{33}$$

Simple calculations give

$$h_{ij,kl} = \frac{x_i x_j x_k x_l}{r^6}[-8 + \frac{3}{1+2\phi} + \frac{3}{(1+2\phi)^2} + \frac{2}{(1+2\phi)^3}] + \frac{1}{r^4}(\delta_{il}x_j x_k + \delta_{jl}x_i x_k +$$

$$\delta_{kl}x_i x_j + \delta_{ik}x_j x_l + \delta_{jk}x_i x_l)[2 - \frac{1}{1+2\phi} - \frac{1}{(1+2\phi)^2}] + \frac{1}{r^2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})[\frac{1}{1+2\phi} - 1]. \tag{34}$$

As in previous Section we find

$$8\pi G t^{st}_{00} = \frac{1}{r^2}[-1 + \frac{2}{1+2\phi} - \frac{1}{(1+2\phi)^2}], \tag{35}$$

$$8\pi G t^{st}_{00}\big|_{\phi\ll 1} = \frac{1}{r^2}[-4\phi^2 + 16\phi^3 + \cdots], \tag{36}$$

and

$$8\pi G t^{st}_{ij} = \frac{x_i x_j}{r^4}[1 - 3\phi - \frac{1}{2(1+2\phi)} - \frac{1}{2(1+2\phi)^2}] + \frac{\delta_{ij}}{r^2}[\phi - \frac{1}{2(1+2\phi)} - \frac{1}{2(1+2\phi)^2}], \tag{37}$$

$$8\pi G t^{st}_{ij}\big|_{\phi\ll 1} = 4\frac{\phi^2}{r^2}\left(\delta_{ij} - \frac{2x_i x_j}{r^2}\right) - 12\frac{\phi^3}{r^2}\left(\delta_{ij} - \frac{5}{3}\frac{x_i x_j}{r^2}\right) + \cdots]. \tag{38}$$

Comparing (36) and (38) with corresponding expressions in harmonic frame, see (12) and (13), we note that there is an essential difference even in ϕ^2 approximation. This means that even $h^{(1)st}$ cannot be obtained from $h^{(1)har}$ by gauge transformation. But the difference in radial coordinates in these systems [see the transition from (5) to (6)], i.e. $r - R = GM$, can't be responsible for the differences in the energy-momentum tensors when $R, r \gg GM$. The blame must be laid upon the violation of the coordinate condition (15), i.e. on the nonphysical degrees of freedom.

Finally, we note that in the considered static field in space without matter the conservation law $t^{\mu\nu}{}_{,\nu}$ takes the form $t_{ij,j} = 0$ outside the horizon. Each term in the expansion of $t_{ij,j}$ in power series of ϕ must satisfy the conservation law and this dictates up to a constant factor the form of the $n-$th term of the expansion:

$$\frac{\phi^n}{r^2}\left(\delta_{ij} - \frac{n+2}{n}\frac{x_i x_j}{r^2}\right).$$

5 Conclusion

We see that the energy-momentum tensor of gravitational field requires the existence of privileged coordinate system and there are some grounds to assume that the rectangular harmonic frame is such a system. Of course, any other frame is also good if we deal with

covariant quantities, but the energy-momentum tensor must be properly transformed from the privileged system. It seems reasonable to expect that in the region of applicability of any theory, its energy-momentum tensor should be finite. More exactly we expect that the total gravitational energy in space outside a radius r must be finite. In general relativity, with definitions of $t_{\mu\nu}$ which seems reasonable, this energy goes to $-\infty$ when $r \to GM$ [9,4]. So the consideration of possible deviations from general relativity are of interest, cf [10].

6 Acknowledgments

This work was supported in part by the Russian Foundation for Basic Research (projects no. 00-15-96566 and 02-02-16944).

References

1. Misner C.W., Thorne K.S., Wheeler J.A., *Gravitation*, Sam Francisco (1973).

2. Weinberg S., *Gravitation and Cosmology*, New Yorc (1972).

3. Schwinger J., *Particle, Sources and Fields*, Addison-Wesley Publishing Company, Reading (1970).

4. Nikishov A.I. gr-qc/0002038; Bulletin of the Lebedev Physics Institute, N12, pp. 45-49 (1999).

5. Fock V., *The Theory of Space-Time and Gravitation.* (2-nd revised edition, Pergamon Press, New York, 1964.)

6. Duff M.J., Phys. Rev. **D**, V. **7**, 2317 (1972).

7. Dehnen H., Hönl H. and Westpfahl, Ann. der Phys. **6**, 7 Folge, Band 6, Heft 7-8, S. 670 (1960).

8. Synge J.L., *Relativity: The General Theory*, Amsterdam (1960).

9. Dehnen H. Zeit. für Phys., **179** Band, 1 Heft, S. 76 (1964).

10. Nikishov A.I., gr-qc/9912034; Part. and Nuclei, **32**,p. 5 (2001).